ENTROPY CONSTRAINED
FRACTAL IMAGE CODING

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Abstract
In this paper we present an entropy constrained fractal coding scheme. In order to get
high compression rates, previous fractal coders used hierarchical coding schemes
with variable range block sizes. Our scheme uses constant range block sizes, but
the complexity of the fractal transformations is adapted to the image contents. The
entropy of the fractal code can be significantly reduced by introducing geometrical
codebooks of variable size and a variable order luminance transformation. We pro-
pose a luminance transformation consisting of a unification of fractal and transform
coding. With this transformation both inter- and intra- block redundancy of an
image can be exploited to get higher coding gain. The coding results obtained with
our new scheme are superior compared to conventional fractal and transform coding
schemes.

1. INTRODUCTION
The principle of fractal image coding consists in finding a construction rule that produces a
fractal image which approximates the original image. Fractal image coding is based on the
mathematical theory of iterated function systems (IFS) developed by Barnsley\(^1\). Jacquin
was the first to propose a block-based fractal image coder\(^2\). This scheme represents the
basis of most published fractal image coding schemes. A detailed introduction to fractal
image coding can be found in\(^3\). An overview of the fractal image compression literature is
given in\(^4\).
Redundancy reduction is achieved by describing the original image through contractively transformed parts of the same image. The image to be encoded is partitioned into non-overlapping range blocks \( f \). For each range block a contractive transformation of a larger block of the same image (a domain block) is determined such that this approximation resembles the range block. A geometrical transformation \( \gamma \) shrinks the domain block and maps it to the position of the range block. The set of allowed geometrical transformations forms a virtual codebook containing the codebook blocks \( g_i \). A luminance transformation \( \lambda \) serves to adjust the intensities of the codebook block (fig. 1). At the decoder the fractal approximation image is constructed by iterating the transformations on an arbitrary initial image.

![Figure 1: Principle of fractal block coding.](image)

In this paper we describe an adaptive luminance transformation that operates in the frequency domain. As this transformation consists of a unification of fractal and transform coding, both intra- and inter-block redundancies of an image can be exploited. Combining this transformation with an entropy constrained selection of the fractal codes a higher compression gain can be obtained. In opposition to previous fractal coders high quality fractal encodings become possible.

2. GEOMETRICAL TRANSFORMATION

The search for a fractal transform can be seen as a search in a codebook that contains the set of contracted domain blocks. Coding efficiency strongly depends on the way in which this codebook is searched. We determined the distribution of codebook block positions that yield the best approximation for a given range block. Very often the best codebook block corresponds to the domain block directly above or close to the position of the range block to be encoded. Nevertheless for some blocks a full search covering the entire image is useful. Increasing the search width generally leads to a better approximation quality. To profit from the distribution of suitable codebook blocks we introduce four search regions relative to the position of the range block as shown in figure 2. Variable costs (amount of bits) for
the geometrical transformation can be achieved using search regions of different sizes. For each search region the codebook block with the lowest approximation error is determined and stored. In a later step of the encoding process the most cost efficient block is selected.

![Geometrical search scheme using four search regions](image)

**Figure 2: Geometrical search scheme using four search regions**

To find the best codebook block of the large and full search region a huge number of codebook blocks has to be examined. Usually the correlation for every range-codebook block pair has to be evaluated. The pair with the highest correlation coefficient \( |\rho| \) represents the best \( \lambda - \gamma \)-pair. Applying a full search leads to very high coding times. To cope with this problem Saupe proposes a fast search method for fractal image coding\(^5\). Range and codebook blocks are seen as vectors in a multidimensional space. The method exploits the fact that similarities between blocks are described by the orientation of these vectors in space. By applying a normalization key-vectors are generated. Now the search can be reduced to a nearest neighbourhood search. Friedman et al. propose an algorithm to find the \( d \) best neighbors in a \( k \)-dimensional space in logarithmic time\(^6\).

The search time for this fast search scheme strongly depends on the dimension of the space to be searched. To reduce the dimension of the key vectors Saupe uses a sub-sampling of the blocks. We obtained reduced coding time by generating lower dimensional key vectors from the low-frequency spectral DCT-coefficients of the blocks.
3. FRACTAL TRANSFORM CODING

3.1 1st order luminance transformation

Jacquin proposed a 1st order luminance transformation. With two parameters: a scaling factor \( a \) and an offset value \( b \) the intensities of a codebook block are adjusted to get an approximation of the range block. These parameters are continuous and therefore have to be quantized.

\[
\hat{f} = \lambda_1(g) = a \cdot g + b
\]

In\(^7\) we introduced a modification of this luminance transformation by scaling only the dynamic part of the codebook block. The mean of the codebook block is scaled with a constant factor \( a_0 \). Choosing a small value for this factor, a better convergence at the decoder is achieved. On the other hand the term \( a_0 \cdot \mu_g \) serves as a prediction of the range block mean. The reduced variance of the \( b \)-values leads to reduced approximation errors with quantized \( a/b \)-parameters. Best coding results were obtained with \( a_0 = 0.5 \). Opposed to the conventional luminance transformation the quantization error components of this transformation are independent of each other.

\[
\hat{f} = \lambda_{1\text{mod}}(g) = a \cdot (g - \mu_g) + 0.5 \cdot \mu_g + b
\]

3.2 High order luminance transformations

Generally large or complexly structured range blocks cannot be approximated well with a 1st order luminance transformation. Approximation quality can be improved by splitting the range blocks into smaller blocks. Using smaller range block sizes leads to an increased number of transformations to be transmitted.

Another way of improving approximation quality is to use higher order luminance transformations. Several proposals to use luminance transformations with additional 1st to 3rd order polynomials have been made\(^8\). The problem that some blocks with fine textures or thin lines cannot be approximated well, however, cannot be solved with these additional low order polynomials.

Vines proposed a fractal coding scheme using an orthogonal basis approach\(^9\). By linearly combining orthonormalized codebook blocks a better approximation quality can be achieved. Due to the contractivity constraints scaling factors larger than 1 cannot be used. Another problem is to assure that the basis functions will be the same at the coder and the decoder. At the coder the basis functions are generated from the original image. At the decoder, however, the basis functions are generated from the reconstruction image. With quantized scaling parameters especially the reconstructed high frequency basis functions at the decoder will differ from those at the coder. The next section describes a scheme to avoid this problem.

3.3 Fractal approximation in the frequency domain

We propose a high order luminance transformation in the frequency domain. To transform range \( f \) and codebook blocks \( g \) of size \( N \cdot N \) pixels to the frequency domain we use the
discrete cosine transform (DCT). The DCT is chosen to profit from the strong energy compaction of this transform. The spectral coefficients of the spectra \( F \) and \( G \) are zigzag-scanned and are referred to as \( F_i \) and \( G_i \). A universal approximation in the frequency domain \( \Lambda \) is given by equation (3). Each spectral coefficient can be changed by an individual scaling factor \( a_i \) and offset an value \( b_i \).

\[
\hat{F} = \Lambda(G) = \\
\begin{bmatrix}
a_0 & 0 & \cdots & 0 \\
0 & a_1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & a_{N^2-1}
\end{bmatrix} \cdot G + \\
\begin{bmatrix}
b_0 \\
b_1 \\
\vdots \\
b_{N^2-1}
\end{bmatrix}
\]

\[
\hat{F} = [\hat{F}_0 \hat{F}_1 \ldots \hat{F}_{N^2-1}]^T = DCT(\hat{f}) \quad \hat{F} = [G_0 G_1 \ldots G_{N^2-1}]^T = DCT(g)
\]

This approximation is overdetermined as \( N \cdot N \) coefficients are approximated with \( 2 \cdot N \cdot N \) parameters. To achieve compression simplified versions of this approximation have to be applied.

The contractivity factor \( s \) of this transformation can be evaluated using the Parseval-theorem. Let \( P \) and \( Q \) be the spectra of any two image blocks.

\[
d(\Lambda(P), \Lambda(Q)) \leq s \cdot d(P, Q)
\]

\[
\sqrt{\sum_{i=0}^{N^2-1} (a_i \cdot P_i + b_i - a_i \cdot Q_i + b_i)^2} \leq s \cdot \sqrt{\sum_{i=0}^{N^2-1} (P_i - Q_i)^2}
\]

(4)

The left side of equation (4) yields:

\[
\sqrt{\sum_{i=0}^{N^2-1} a_i^2 \cdot (P_i - Q_i)^2} \leq \max(|a_i|) \cdot \sqrt{\sum_{i=0}^{N^2-1} (P_i - Q_i)^2}
\]

\[
\Rightarrow s = \max(|a_i|)
\]

(5)

The contractivity of the transformation \( \Lambda \) only depends on the scaling factors \( a_i \). A perfect approximation of any range block spectrum is possible by setting the additive components \( b_i \) to the spectral weights of \( F \).

### 3.4 Fractal Transform Coding

Fractal coding exploits inter-block similarities between different scales of the image, whereas transform coding reduces intra-block redundancy.

We propose fractal transform coding (FTC) as a unification of fractal and transform coding. FTC uses a combined approximation method thus exploiting intra- and inter-block redundancies of image blocks. The major part of a range block spectrum is approximated using the fractal transform. Only those spectral coefficients that cannot be approximated well by the fractal transform are excluded from the fractal approximation. These spectral coefficients are individually coded as with transform coding. Using this combined approximation the entropy of the luminance transformation can be significantly reduced compared
<table>
<thead>
<tr>
<th>Coding Scheme:</th>
<th>FBC (Fractal Block Coding)</th>
<th>FTC (Fractal Transform Coding)</th>
<th>TC (Transform Coding)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Approximation of &quot;Range&quot; Blocks</td>
<td>( \tilde{F} = \begin{bmatrix} a \cdot G_0 \ a \cdot G_1 \ \vdots \ a \cdot G_{N^2-1} \end{bmatrix} + \begin{bmatrix} b_0 \ 0 \ 0 \ \vdots \ 0 \end{bmatrix} )</td>
<td>( \tilde{F} = \begin{bmatrix} 0.5 \cdot G_0 \ 0 \ \vdots \ a \cdot G_{N^2-1} \end{bmatrix} + \begin{bmatrix} b_0 \ b_1 \ 0 \ \vdots \ 0 \end{bmatrix} )</td>
<td>( \tilde{F} = \begin{bmatrix} b_0 \ b_1 \ \vdots \ b_{N^2-1} \end{bmatrix} )</td>
</tr>
</tbody>
</table>

![Fractal Block Coding Diagram](image)

Figure 3: Comparison of the approximation schemes of fractal block coding, fractal transform coding, and transform coding. Fractal block coding uses a scaled, mean adjusted codebook block to approximate a range block. Transform coding uses no scaling of a codebook block. The range block spectrum is approximated with the (quantized) spectral weights of the range block. Fractal transform coding unifies these two principles. The major part of a spectrum is approximated with a fractal transform, only those spectral coefficients that cannot be approximated well, are approximated as with transform coding. (An example of a 3rd order FTC approximation is shown.)

to transform coding. In many cases good approximations can be achieved using only very few individual coded spectral coefficients. Fig. 3. compares the approximation schemes of fractal block coding, transform coding, and fractal transform coding.

An approximation with \( K \) individually coded dynamic spectral coefficients can be seen as a luminance transformation of order \( K + 1 \). The approximation scheme using FTC is given by:

\[
\tilde{F} = \Lambda_K(G) = 0.5 \cdot M_0 \cdot G + b_0 \cdot B_0 + a \cdot M_k \cdot G + M_k^* \cdot F_q
\]

\[ M_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}; \quad B_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}; \]
The matrix $M_K$ masks the region $R$ of those dynamic spectral coefficients that are approximated by the fractal transform. For the rest of the dynamic spectrum, which is masked by $M_K^*$, the scaling factors $a_i$ are set to 0 while the $b_i$-values are set to the (quantized) spectral weights of the range block $F_q$. As these spectral coefficients are excluded from the fractal approximation, a better approximation can be achieved for the rest of the spectrum.

Using the FTC with quantized parameters, the approximation error of the range block spectrum consists of three parts: the approximation error of the mean, the error of the fractal approximation of the dynamic spectrum (region $R$), and the quantization error of the TC-coded spectral coefficients:

$$
e^2 = (f_0 - (0.5 \cdot G_0 + b_0))^2 + \sum_{l \in R} (F_l - a \cdot G_l)^2 + \sum_{l \notin R} (F_l - b_l)^2$$

$$e^2 = \Delta b_0^2 + \sum_{l \in R} \Delta a_l^2 \cdot G^2_l + \sum_{l \notin R} \Delta b_l^2$$

with $\Delta b_0 = F_0 - 0.5 \cdot G_0$; $\Delta a_l = \frac{F_l}{G_l} - a$; $\Delta b_l = F_l - b_l$

The optimal transformation parameters can be found as:

$$b_{0\text{opt}} = f_0 - 0.5 \cdot G_0; \quad b_{l \text{opt}} = F_l; \quad a_{\text{opt}} = \frac{\sum_{l \in R} F_l \cdot G_l}{\sum_{l \in R} G^2_l}$$

To achieve high compression the order of the FTC-transformation has to be adapted to the range block to be encoded. When increasing the order of a FTC-luminance transformation from $K$ to $K+1$ an extra spectral coefficient has to be determined. The best new additional spectral coefficient with index $t$ to be encoded with TC is the one that maximizes the error reduction:

$$\max(e^2 - e^2(t)) \quad \text{with} \quad t \in R$$

$$= \max\left(\Delta b_0^2 - \Delta b_0^2 + \Delta a_t^2 \cdot G_t^2 - \Delta b_t^2 + \sum_{l \in R \setminus t} (\Delta a_l^2 - \Delta a_t^2) \cdot G_t^2\right)$$

4. ENTROPY CONSTRAINED FRACTAL CODING

To achieve high compression the expense for the approximations has to be adapted to the contents of the range blocks. For simply structured blocks very often a good approximation can be achieved using a 1st order luminance transformation and a small geometrical codebook. For other blocks however, good approximations are only possible with a larger
geometrical codebook and a higher order luminance transformation. By introducing approximation methods of variable complexity, additional classification information has to be transmitted to the decoder. Nevertheless the total entropy of the fractal code can be strongly reduced.

Determining the — in the rate-distortion-sense — optimal fractal code for a given image is very computational expensive. In\textsuperscript{10} we introduced a sub-optimal method for selecting the order of geometrical and luminance transformation for each range block. Starting with a fractal code using a 0 bit geometrical codebook and 1\textsuperscript{st} order luminance transformation for all range blocks, the fractal code is successively changed in such a way that at each step the most cost efficient improvement is obtained. Figure 4, compares the coding results of an entropy constrained coder to a conventional fractal coding scheme using one geometrical codebook and 1\textsuperscript{st} order luminance transformation. It can be seen that for a given approximation quality the rate can be significantly reduced with the entropy constrained coding scheme.

Figure 4: Comparison of coding results using normal and entropy constrained fractal coding schemes:

(A) one geometrical codebook using a 1\textsuperscript{st} order luminance transformation,
(B) entropy constrained selection of the geometrical codebook (sizes 0, 4 and 14 bits) using a 1\textsuperscript{st} order luminance transformation, and
(C) entropy constrained selection of the geometrical codebook and the order of the FTC-luminance transformation.

The numbers indicate the size of the range blocks (4 × 4 and 8 × 8 pixels).
5. RESULTS AND CONCLUSION

Figures 5 and 6 compare the coding results of the new scheme to a quadtree-based fractal coder, a fractal coder using HV-partitions, and the JPEG standard. It can be seen that the entropy constrained coding scheme using the FTC outperforms the other coders. Future research will concentrate on finding an optimal selection of the fractal codes. Another aspect is to reduce the computational complexity of the new scheme.

Figure 5: Comparison of coding results for the “Lena” image (512 x 512 pixel).

Figure 6: Comparison of coding results for the “Boat” image (512 x 512 pixel).
6. REFERENCES